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ON THE STRUCTURE OF SPACES REPRESENTING A LANDWEBER EXACT COHOMOLOGY THEORY

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INTRODUCTION

RECALL that a cohomology theory $E^*(-)$ is *complex oriented* if there is a class $x \in E^2(\mathbb{C}P^\infty)$ such that $E^*(S^2)$ is generated as a free module over E^* (point) by $i^*(x)$, where $i: S^2 \to \mathbb{C}P^\infty$ is the inclusion of the bottom cell. This property is enough to ensure a natural transformation $\omega: MU^*(-) \to E^*(-)$, where $MU^*(-)$ is complex cobordism. In the case of the theory $E^*(-)$ possessing a compatible product, Landweber [7] gave criteria on the structure of the coefficient ring, $E^*(\text{point}) = E^* = E_{-*}$ say, which ensures a simple relationship between $MU^*(X)$ and $E^*(X)$. This is best stated in the dual, homology variant as the tensor product

$$E_*(X) = E_* \otimes_{MU_*} MU_*(X),$$

holding for any space X, the MU_*-action on E_* being that induced from the complex orientation ω.

There are many important examples of such Landweber exact theories. Complex K-theory, the Brown–Peterson theories $BP_*(-)$ and those derived from the Johnson–Wilson spectra $E(n)$ have been known for some time. More recently, much interest has centred on a new example, that of elliptic homology [8], and we have reason to hope that this is just the beginning of a whole series of global and analytically interesting theories, one corrresponding to each chromatic type; for more details see the first author's paper [4].

Meanwhile, Brown [3] has shown that in suitable circumstances (certainly always satisfied by the theories considered in this paper), cohomology theories are *representable*. Recall that $E^*(-)$ is representable if there are spaces $\mathbf{E}_*$ with natural isomorphisms

$$E^r(-) \cong [-, \mathbf{E}_r],$$

where the right-hand side indicates the set of homotopy classes of maps into the representing space $\mathbf{E}_r$. We shall use the convention of writing $\mathbf{E}'_r$ to indicate the connected component of the basepoint of $\mathbf{E}_r$.

The following work gives an analysis of both the geometric and the homological structure of the spaces representing a Landweber exact cohomology theory. Geometrically, we prove (Theorem 4.1) that after localisation at some prime p, such spaces can be decomposed as colimits of finite products of Wilson spaces: recall that Wilson [10] showed for each prime p and positive integer s the existence of essentially unique $s-1$ connected H-spaces $Y_s = Y_s(p)$ with $\pi_*(Y_s)$ and $H_*(Y_s; \mathbb{Z}_{(p)})$ free over $\mathbb{Z}_{(p)}$ and $\pi_s(Y_s) = \mathbb{Z}_{(p)}$. In fact, these spaces all occur in the Ω-spectra for $BP\langle n\rangle$ for various n.

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From this a number of other results follow: for example, the homologies of such spaces $\mathbf{E}_*$ are torsion free. Moreover, the $\mathbb{Z}_{(p)}$ homology of each even-graded space $\mathbf{E}'_{2r}$ is polynomial, while that of the odd-graded space $\mathbf{E}'_{2r+1}$ is exterior on the suspensions of the generators of $H_*(\mathbf{E}_{2r}; \mathbb{Z}_{(p)})$; the product here is that derived from the loop-space structure of these spaces. However, to present our full results on the homology structure we need the language of *Hopf rings*.

2. HOPF RINGS

Let us write $H_*(X;S)$ for the ordinary homology of the space X with coefficients in the commutative ring S. Suppose we have a multiplicative Ω-spectrum $\{\mathbf{E}_r; r \in \mathbb{Z}\}$ together with a Künneth isomorphism $H_*(\mathbf{E}_r \times \mathbf{E}_r; S) \cong H_*(\mathbf{E}_r; S) \otimes H_*(\mathbf{E}_r; S)$ for each (the tensor product here and, unless otherwise indicated, below being taken over the coefficients S). Then the H-space structure of $\mathbf{E}_r$ makes $H_*(\mathbf{E}_r; S)$ a graded Hopf algebra. However, the work of Ravenel and Wilson [9] shows that the study of the homology of the individual spaces $\mathbf{E}_r$ is best carried out by studying the global bigraded algebraic object $H_*(\mathbf{E}_*; S)$. This is largely due to the existence of a product

$$\circ : H_*(\mathbf{E}_r; S) \otimes H_*(\mathbf{E}_s; S) \to H_*(\mathbf{E}_{r+s}; S)$$

induced from the multiplication in E-theory. This $\circ$-product together with the loop-space structures on the $\mathbf{E}_r$ makes $H_*(\mathbf{E}_*; S)$ into a graded ring object in the category of graded coalgebras: an algebraic system known as a *Hopf ring* (see [9] for details).

To set up some notation, we denote the H-space product $H_*(\mathbf{E}_r; S) \otimes H_*(\mathbf{E}_r; S) \to H_*(\mathbf{E}_r; S)$ by the symbol $*$, and write ψ for the corresponding coproduct. Certain elements are easily constructed: for each homogeneous element $v \in E_*$ we obtain a Hopf ring element $[v] \in H_0(\mathbf{E}_{-|v|}; S)$ as follows. If $v \in \pi_{-r}(E)$, regard it as an unbased map

$$v : \text{point} \to \mathbf{E}_r.$$

Now take homology (all homology in this paper is unreduced) and define $[v]$ as the image under v_* of a selected generator of $H_0(\text{point}; S) = S$. This class of Hopf ring elements enjoy the properties

$$[v] * [w] = [v + w]$$
$$[v] \circ [w] = [vw]$$
$$\psi[v] = [v] \otimes [v].$$

The set of elements of this form generate a sub-Hopf ring of $H_*(\mathbf{E}_*; S)$ which we denote by $S[E^*]$. In fact, this latter Hopf ring is exactly the zero-dimensional part of $H_*(\mathbf{E}_*; S)$.

If $E^*(-)$ is complex oriented, then the orientation corresponds to a map

$$\omega_E : \mathbb{C}P^\infty \to \mathbf{E}_2.$$

Let us take $\{\beta_t; t = 0, 1, 2, \dots, \beta_t \in H_{2t}(\mathbb{C}P^\infty; S)\}$ as the dual basis for $H_*(\mathbb{C}P^\infty; S)$ corresponding to the powers of ω_H in $H^*(\mathbb{C}P^\infty; S)$ and define $b_t \in H_{2t}(\mathbf{E}_2; S)$ as $(\omega_E)_*(\beta_t)$. When we work with mod p coefficients, it is usual to employ the notation $b_{(s)}$ for the element b_{p^s}. We note that the element b_1 may be written as $e_1 \circ e_1$ where $e_1 \in H_1(\mathbf{E}_1; S)$ is the homology suspension of [1] (the Hopf ring element, not the reference). The element [1] itself is identical to b_0. The homology coproduct satisfies the formula

$$\psi(b_t) = \sum_{i=0}^{t} b_i \otimes b_{t-i}$$

and e_1 is primitive.

Following [9], we denote by $H_*^R(\mathbf{E}_*; S)$ the free $S[E^*]$ Hopf ring on generators $b_t \in H_{2t}(\mathbf{E}_2; S)$ and the suspension element $e_1 \in H_1(\mathbf{E}_1; S)$, modulo the formal group law relations [9, (3.8)] and the identities $e_1 \circ e_1 = b_1$ and $[1] = b_0$. There is a natural Hopf ring map

$$\tau_E : H_*^R(\mathbf{E}_*; S) \to H_*(\mathbf{E}_*; S).$$

The results of [9] calculate the Hopf rings for MU and BP-theories. It is shown that the Hopf algebras $H_*(\mathbf{MU}_r'; \mathbb{Z})$ are torsion free, while $H_*(\mathbf{MU}_r'; \mathbb{Z}_{(p)})$ is bipolynomial for r even and exterior for r odd. The Hopf ring $H_*(\mathbf{MU}_*; \mathbb{Z})$ is shown to be isomorphic to $H_*^R(\mathbf{MU}_r; \mathbb{Z})$ via τ_{MU}; similar results hold for BP.

We encapsulate these properties in the following definition.

Definition 2.1. Let us say that a multiplicative, complex-oriented spectrum E is $\mathscr{RW}$ if it satisfies the following three conditions:

(i) $H_*(\mathbf{E}_r'; \mathbb{Z})$ is torsion free for each r.
(ii) $H_*(\mathbf{E}_r'; \mathbb{Z}_{(p)})$ is an exterior algebra for r odd and is polynomial for r even,
(iii) the Hopf ring map $\tau_E : H_*^R(\mathbf{E}_*; \mathbb{Z}) \to H_*(\mathbf{E}_*; \mathbb{Z})$ is an isomorphism.

We can now state our main result on the homology of spaces representing a Landweber exact theory.

THEOREM 2.2. *Let the Ω-spectrum $\mathbf{E}_*$ represent a Landweber exact multiplicative cohomology theory. If the coefficients E_* are concentrated in even dimensions and are a free R-module of countable rank for some subring R of $\mathbb{Q}$, then E is $\mathscr{RW}$.*

Remark 2.3. As we claim that $H_*(\mathbf{E}_r; \mathbb{Z})$ is torsion free for all r we have the Künneth isomorphism $H_*(\mathbf{E}_r \times \mathbf{E}_r; \mathbb{Z}) \cong H_*(\mathbf{E}_r; \mathbb{Z}) \otimes H_*(\mathbf{E}_r; \mathbb{Z})$. Thus, $H_*(\mathbf{E}_*; \mathbb{Z})$ is indeed a Hopf ring.

Remark 2.4. Landweber's criteria [7] for a theory to be exact over MU ensure that we can always assume the coefficients E_* themselves to be torsion free.

Remark 2.5. It is immediate from the definition that we have the relationship

$$H_*^R(\mathbf{E}_*; \mathbb{Z}) = H_*^R(\mathbf{MU}_*; \mathbb{Z}) \otimes_{\mathbb{Z}[MU^*]} \mathbb{Z}[E^*].$$

Thus, the corresponding result for MU plus the results of the above theorem show there to be an equivalence of Hopf rings

$$H_*(\mathbf{E}_*; \mathbb{Z}) \cong H_*(\mathbf{MU}_*; \mathbb{Z}) \otimes_{\mathbb{Z}[MU^*]} \mathbb{Z}[E^*]$$

for Landweber exact E.

Remark 2.6. The theorem ensures the collapsing of the Atiyah–Hirzebruch spectral sequence for $F_*(\mathbf{E}_r)$ for an arbitrary multiplicative spectrum F. Moreover, if F is complex oriented as well, the formulae for the homology coproduct plus the formal group law relations for the interaction of E and F allow us to write down the Hopf ring $F_*(\mathbf{E}_*)$ for such theories. We obtain the relation

$$F_*(\mathbf{E}_*) = F_*(\mathbf{MU}_*) \otimes_{F_*[MU^*]} F_*[E^*].$$

See, for example, the proofs of 4.5 and 4.7 of [9]. As a further corollary we have a description of the unstable cooperation algebra: just as the exactness property allows us to write the stable cooperations as

$$E_*(E) = E_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} E_*$$

the above equivalence gives the unstable analogue

$$E_*(\mathbf{E}_*) = E_* \otimes_{MU_*} MU_*(\mathbf{MU}_*) \otimes_{MU_*[MU^*]} MU_*[E^*].$$

Remark 2.7. The hypotheses of the theorem are all satisfied by the standard examples of Landweber exact theories: BP, KU, Ell, $E(n)$, etc. The completed spectra $\widehat{E(n)}$ of Baker and Würgler [2] also give rise to Landweber exact cohomology theories, but fail the hypotheses, their coefficient rings not being free over any R. The homologies of the spaces $\widehat{\mathbf{E(n)}}_*$ are discussed in detail in [1].

Our work uses the Hopf ring calculation of MU in [9] plus consequences of the Landweber exactness property. As noted above, the result for BP was already included in [9], while that for $E(n)$ was described by the second author in his Ph.D. thesis and again in [5] using an alternative method which rested on the BP result and Wilson's splitting theorem [10]. An alternative, and perhaps simpler, algebraic description of the Hopf rings $F_*(\mathbf{E}_*)$ for Landweber exact E comes from the $F_*^{\mathcal{Q}}(\mathbf{E}_*)$ construction of [6]. The reader is directed to that article for details of this object and its relationship to the work here.

3. PRELIMINARY CONSTRUCTIONS

From now on $E^*(-)$ shall always denote a Landweber exact cohomology theory (with representing Ω-spectrum $\mathbf{E}_*$) which satisfies the hypotheses outlined in the statement of Theorem 2.2.

The key to the proof of our results is the consideration of the homology theory $E_* \otimes_{\mathbb{Z}} MU_*(-)$. This theory has an associated Ω-spectrum $\mathbf{M}_*$ and likewise there is a map of spectra $f: \mathbf{M}_* \to \mathbf{E}_*$ associated with the homomorphism

$$f_*: M_*(-) = E_* \otimes_{\mathbb{Z}} MU_*(-) \to E_* \otimes_{MU_*} MU_*(-) = E_*(-).$$

PROPOSITION 3.1. (a) *The spectrum M is $\mathcal{RW}$.*

(b) *The induced map $f_*: H_*(\mathbf{M}_r; A) \to H_*(\mathbf{E}_r; A)$ is onto for any coefficients A.*

Remark 3.2. This proposition is the central observation for our result as it provides us with a theory, M, whose Hopf ring has the properties we desire and also covers (in the sense of part (b)) the Hopf ring for the Landweber exact theory we are interested in. We point to the analogous part of [9], namely the start of Section 4, where the Hopf ring for BP is deduced from that for MU–in the case of BP just one copy of MU is needed (essentially Quillen's theorem); we need the above more complex result to handle an arbitrary exact spectrum as the homomorphism $MU_* \to E_*$ will not in general be epimorphic.

Proof of Proposition 3.1. We begin by giving an explicit description of the spaces $\mathbf{M}_r$. Recalling the assumptions on E_*, suppose that $\mathscr{E}$ is a countable set of R-module generators for E_*, and that we write $\mathscr{E} = \operatorname{colim}_{\alpha \in \mathbb{Z}} \mathscr{E}_\alpha$, where each $\mathscr{E}_\alpha$ is a finite set. Then we readily check that

$$\operatorname*{colim}_{\alpha \in \mathbb{Z}} \prod_{e \in \mathscr{E}_\alpha} \mathbf{MUR}_{r-|e|}.$$

is the rth space of an Ω-spectrum representing M: we take this as a model for $\mathbf{M}_r$.

Note that the results of [9] and Section 4 show the spectrum MUR also to be $\mathscr{RW}$; in particular, we may write, as Hopf rings over any coefficients A,

$$H_*(\mathbf{MUR}_*; A) = A[R] \otimes [A\mathbb{Z}] H_*(\mathbf{MU}_*; A)$$

That $H_*(\mathbf{M}_r; \mathbb{Z})$ is torsion free and $H_*(\mathbf{M}'_r; \mathbb{F}_p)$ is polynomial (exterior) follows directly from the above description of $\mathbf{M}_r$ in terms of the spaces in the Ω-spectrum for MUR. For the result about τ_M we have

$$\begin{aligned} H^R_*(\mathbf{M}_*; \mathbb{Z}) &= \mathbb{Z}[E^*] \otimes_{Z[R]} H^R_*(\mathbf{MUR}_*) \\ &= \mathbb{Z}[E^*] \otimes_{Z[R]} H_*(\mathbf{MUR}_*) \\ &= H_*(\mathbf{M}_*; \mathbb{Z}) \end{aligned}$$

where the first equality is by the construction of $H^R_*(-; \mathbb{Z})$, the second by the observation that MUR is $\mathscr{RW}$, and the third by the construction of the spaces $\mathbf{M}_r$. This proves part (a).

For part (b) we note that the homomorphism $f_*: M_*(-) \to E_*(-)$ is always surjective. Thus, by Spanier–Whitehead duality, the induced map on cohomology theories,

$$f_*: [-, \mathbf{M}_r] \to [-, \mathbf{E}_r]$$

is surjective on finite complexes. Now, suppose we write $\mathbf{E}_r = \operatorname{colim} X_\alpha$ for some set of finite complexes X_α. We can lift each $X_\alpha \to \mathbf{E}_r$ in the directed system to a map $X_\alpha \to \mathbf{M}_r$. As any element in $H_*(\mathbf{E}_r; A)$ is in the image of the homomorphism from $H_*(X_\alpha; A)$ for some α, the result is proved. □

4. THE GEOMETRIC CONSEQUENCES

Let us recall the work of Wilson [10]. There the spaces in the Ω-spectrum for BP at some given prime p are decomposed as follows: for $s \leq 2(p^n + \cdots + p + 1)$,

$$\mathbf{BP}_s \simeq \mathbf{BP}\langle\mathbf{n}\rangle_s \times \prod_{j>n} \mathbf{BP}\langle\mathbf{j}\rangle_{s+2(p^j-1)}$$

and for $s > 2(p^{n-1} + \cdots + p + 1)$ this is as irreducibles; if $s < 2(p^n + \cdots + p + 1)$ it is as H-spaces. Following Wilson, we write Y_s for $\mathbf{BP}\langle\mathbf{n}\rangle_s$ if $2(p^n + \cdots + p + 1) \geq s > 2(p^{n-1} + \cdots + p + 1)$. This splitting result plus the structure of the Hopf ring for BP shows that $H_*(\Omega^2 Y_s; \mathbb{Z}_{(p)})$ is (bi)polynomial for s even and exterior for s odd.

THEOREM 4.1. *If the Ω-spectrum $\mathbf{E}_*$ represents a Landweber exact cohomology theory with coefficients E_* concentrated in even dimensions and free as an R-module of countable rank for some subring R of $\mathbb{Q}$, then the p-localised spaces $(\mathbf{E}'_r)_{(p)}$ can be written as a colimit of finite products of the Y_s.*

Proof. We have already seen that the spaces $\mathbf{E}'_r$ have torsion-free homotopy. In fact, their homology is torsion free too: as $H_*(\mathbf{E}_r; \mathbb{Z})$ is torsion free, the existence of torsion, say p torsion, in $H_*(\mathbf{E}_r; \mathbb{Z})$ would contradict the statement of Proposition 3.1 (b) for $A = \mathbb{F}_p$.

Wilson [10, (6.2)] shows that any p-local, locally finite H-space with both homotopy and homology torsion free can be written as a product of the Y_s. However, in general the $(\mathbf{E}'_r)_{(p)}$ are not locally finite; we must check that the proof of [10, (6.2)] can be extended. One readily finds that the maps f_s in that proof can be replaced by the ones mapping to a colimit of finite products of Y_s, and thence we get a map $(\mathbf{E}'_r)_{(p)} \to \operatorname{colim}_t (\prod_{s \leq t} (\operatorname{colim}_i \mathrm{Y}^i_s))$ i.e., from $(\mathbf{E}'_r)_{(p)}$ to a colimit of finite products of Wilson spaces. This map is analogous to the $f: X \to Y$ of [10, (6.2)], and the corresponding proof shows our map also to be a homotopy equivalence. □

This result has some immediate consequences for the homology of the spaces $\mathbf{E}_r$.

COROLLARY 4.2. (a) $H_*(\mathbf{E}_r; \mathbb{Z})$ *is torsion free,*
(b) $H_*(\mathbf{E}'_{2r}; \mathbb{Z}_{(p)})$ *is a polynomial algebra,*
(c) $H_*(\mathbf{E}'_{2r+1}; \mathbb{Z}_{(p)})$ *is an exterior algebra.*

Proof. Part (a) was demonstrated in the proof of (4.1). For part (b), the construction in Theorem 4.1 after a double looping gives us an equivalence, now of H-spaces,

$$(\mathbf{E}_{2r-2})_{(p)} \simeq \operatorname{colim} \Omega^2 B_{s,p}$$

where each $B_{s,p}$ is a finite product of Wilson spaces at the prime p. Thus, we see that each $H_*(\mathbf{E}'_{2r}; \mathbb{Z}_{(p)}) = H_*((\mathbf{E}'_{2r})_{(p)}; \mathbb{Z}_{(p)})$ is polynomial over $\mathbb{Z}_{(p)}$ since the directed system is of inclusions of factors in finite products of double-looped Wilson spaces. Part (c) follows similarly. $\square$

Remark 4.3. As $H_*(\mathbf{MU}_{2r-1}; \mathbb{F}_p)$ is an exterior algebra, the dual Rothenberg–Steenrod spectral sequence

$$\{\operatorname{Tor}^{H_*(\mathbf{MU}_{2r-1}; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)\}^* \Rightarrow H^*(\mathbf{MU}'_{2r}; \mathbb{F}_p)$$

has $E_2 = E_\infty$ term a polynomial algebra; this shows $H^*(\mathbf{MU}'_{2r}; \mathbb{F}_p)$ also to be polynomial. It does not follow from our calculations, however, that $H^*(\mathbf{E}'_{2r}; \mathbb{F}_p)$ is similarly polynomial as, in general, the algebras involved are no longer of finite type. Indeed, the cohomology rings for these spaces will usually be far larger than polynomial.

5. THE HOPF RING STRUCTURE

In this section we complete the proof of Theorem 2.2 by demonstrating the following result

THEOREM 5.1. *For E as in Theorem* 2.2, $\tau_E: H^R_*(\mathbf{E}_*; \mathbb{Z}) \to H_*(\mathbf{E}_*; \mathbb{Z})$ *is an isomorphism of Hopf rings.*

Proof. To see that τ_E is surjective we note the following commutative diagram:

$$\begin{array}{ccc} H^R_*(\mathbf{M}_*; \mathbb{Z}) & \xrightarrow{f^R_*} & H^R_*(\mathbf{E}_*; \mathbb{Z}) \\ \cong \downarrow \tau_M & & \downarrow \tau_E \\ H_*(\mathbf{M}_*; \mathbb{Z}) & \xrightarrow{f_*} & H_*(\mathbf{E}_*; \mathbb{Z}). \end{array}$$

By Proposition 3.1(a) τ_M is an isomorphism, while (b) stated that f_* was surjective; we deduce that τ_E is surjective as well. Our job here is to prove that τ_E is injective.

We begin by making several reductions of the problem. Firstly, it will suffice to prove the result after localisation at each prime p; moreover, as $H_*(\mathbf{E}'_r; \mathbb{Z})$ is torsion free, we need only work with $\operatorname{mod} p$ coefficients. Secondly, recall the above commutative diagram. As $H_*(\mathbf{M}'_r; \mathbb{F}_p)$ and $H_*(\mathbf{E}'_r; \mathbb{F}_p)$ are free-graded commutative algebras, it suffices to show that τ_E induces an injection on indecomposables, $Q\tau_E: QH^R_*(\mathbf{E}_*; \mathbb{F}_p) \to QH_*(\mathbf{E}_*; \mathbb{F}_p)$ say.

For ease of notation, let us identify $H^R_*(\mathbf{M}_*; \mathbb{F}_p)$ and $H_*(\mathbf{M}_*; \mathbb{F}_p)$ via the isomorphism τ_M. Then, to prove Theorem 5.1, we need to show that $\ker Qf_* = \ker Qf^R_*$ (note that f^R_* is surjective by construction). In particular, we must show $\ker Qf_* \subset \ker Qf^R_*$ as the opposite inclusion is immediate from the diagram.

To do this we shall first consider the map f_* on degree zero indecomposables. Let

$$K_r = \ker\{QH_0(\mathbf{M}_r; \mathbb{F}_p) \to QH_0(\mathbf{E}_r; \mathbb{F}_p)\}.$$

LEMMA 5.2. *We can find bases for* $\ker\{Qf_*: QH_*(\mathbf{M}'_*; \mathbb{F}_p) \to QH_*(\mathbf{E}'_r; \mathbb{F}_p)\}$ *from the* $\circ$*-ideal generated by* K_*.

This will suffice to prove the theorem as all elements of the form $v \circ \omega \in H_*^R(\mathbf{M}_r; \mathbb{F}_p)$, $v \in K_*$, clearly lie in the kernal of f_*^R.

Proof of Lemma 5.2. We prove the lemma by induction on the homology degree, t say. There is nothing to prove for the case of $t = 0$.

We shall construct disjoint sets of elements $\mathscr{I}_r$ and $\mathscr{K}_r$ in $H_*(\mathbf{M}'_r; \mathbb{F}_p)$ with the following properties:

(a) that their union projects to a basis of $QH_*(\mathbf{M}'_r; \mathbb{F}_p)$,
(b) that the projection of $\mathscr{K}_r$ lies in the $\circ$-ideal generated by K_* and is a basis of

$$\ker\{Qf_*: QH_*(\mathbf{M}'_r; \mathbb{F}_p) \to QH_*(\mathbf{E}'_r; \mathbb{F}_p)\}.$$

Then clearly $\mathscr{I}_r$ maps (isomorphically) via f_* to a set of elements in $H_*(\mathbf{E}'_r; \mathbb{F}_p)$ which project to a basis of $QH_*(\mathbf{E}'_r; \mathbb{F}_p)$. For our inductive argument, let us write $\mathscr{I}_{r,t}$ for $\{z \in \mathscr{I}_r \mid \dim z = t\}$ and similarly $\mathscr{K}_{r,t}$ for $\{z \in \mathscr{K}_r \mid \dim z = t\}$.

The key to the induction is the Rothenberg–Steenrod spectral sequence (RSSS): for any Ω-spectrum $\{\mathbf{G}_r; r \in \mathbb{Z}\}$ we have the relation $\Omega\mathbf{G}_{r+1} = \mathbf{G}_r$ and hence, taking classifying spaces, we have a spectral sequence

$$E^2 = \mathrm{Tor}^{H_*(\mathbf{G}_r; \mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \;\Rightarrow\; H_*(\mathbf{G}'_{r+1}; \mathbb{F}_p).$$

The important point to note is that knowledge of $H_*(\mathbf{G}_r; \mathbb{F}_p)$ for $* < t$ gives rise to knowledge of $H_*(\mathbf{G}_{r+1}; \mathbb{F}_p)$ in degrees $* < t + 1$.

In both the cases $\mathbf{G}_r = \mathbf{M}_r$ and $\mathbf{G}_r = \mathbf{E}_r$ these spectral sequences always collapse: for r odd the E^2-terms are all even dimensional, while for r even the E^2-terms are exterior algebras with generators in Tor_1 (recall that in this spectral sequence, differentials lower Tor-degree).

For a graded set $\mathscr{Z}$, let us write $\mathscr{F}(\mathscr{Z})$ for the free-graded commutative algebra over $\mathbb{F}_p$ on $\mathscr{Z}$. Then the map f gives rise to a morphism of E^∞-terms of the form (omitting terms from π_0)

$$\mathrm{Tor}^{\mathscr{F}(\mathscr{I}_r)} \otimes \mathrm{Tor}^{\mathscr{F}(\mathscr{K}_r)} \to \mathrm{Tor}^{\mathscr{F}(\mathscr{I}_r)}$$

acting as projection of the first factor.

To start the induction recall that, for an Ω-spectrum $\{\mathbf{G}_r; r \in \mathbb{Z}\}$, $H_0(\mathbf{G}_r; \mathbb{F}_p)$ is just the group-ring $\mathbb{F}_p\{\pi_0(\mathbf{G}_r)\}$. Pick a set $\mathscr{K}_{r,0} \subset H_*(\mathbf{M}_r; \mathbb{F}_p)$ projecting to a basis for K_r and extend it via a set $\mathscr{I}_{r,0}$ so as to project to a basis for the whole of $QH_0(\mathbf{M}_r; \mathbb{F}_p)$. An application of the RSSSs for $H_*(\mathbf{M}'_{r+1}; \mathbb{F}_p)$ and $H_*(\mathbf{E}'_{r+1}; \mathbb{F}_p)$, $* \le 1$, is trivial for r odd, while for r even it gives sets

$$\mathscr{I}_{r,1} = \{v \circ e_1 \mid v \in \mathscr{I}_{r,0}\} \quad \text{and} \quad \mathscr{K}_{r,1} = \{v \circ e_1 \mid v \in \mathscr{K}_{r,0}\}$$

satisfying the demands of the construction in the dimensions considered.

Suppose that we have constructed $\mathscr{I}_{r,s}$ and $\mathscr{K}_{r,s}$ for all $s \le t$ and all r. If r is even, the RSSS shows that $H_*(\mathbf{M}'_{r+1}; \mathbb{F}_p)$ and $H_*(\mathbf{E}'_{r+1}; \mathbb{F}_p)$ are exterior algebras on generators the suspensions of generators of $H_*(\mathbf{M}'_r; \mathbb{F}_p)$ and $H_*(\mathbf{E}'_r; \mathbb{F}_p)$, respectively, suspension raising homology dimension by 1. Then we can take $\mathscr{I}_{r+1,t+1} = \{z \circ e_1 \mid z \in \mathscr{I}_{r,t}\}$ and

$\mathscr{K}_{r+1,t+1} = \{z \circ e_1 \mid z \in \mathscr{K}_{r,t}\}$. These sets have the required properties by the result for $\mathscr{I}_{r,t}$ and $\mathscr{K}_{r,t}$.

If r is odd, the E^∞-page of the RSSS is a divided power algebra on the suspension of the generators of $H_*(\mathbf{G}_r; \mathbb{F}_p)$, $\mathbf{G} = \mathbf{M}$ or $\mathbf{E}$. The argument of [9] (see foot of p. 269) then shows $H_*(\mathbf{M}'_{r+1}; \mathbb{F}_p)$ to have an $\mathbb{F}_p$-basis given by monomials $z_1^{*a_1} * \cdots * z_l^{*a_l}$, where $1 \leq a_i < p$ and each $z_i = \sigma^{k(j)}(z)$ for some $z \in \mathscr{I}_r \cup \mathscr{K}_r$ and $k(j) \in \{0, 1, 2, \ldots\}$, the pairs $(k(j), z)$ all being distinct. Here we are defining σ^k by

$$\sigma^k(v \circ e_1 \circ b_{(0)}^{j0} \circ b_{(1)}^{j1} \circ b_{(2)}^{j2} \circ \cdots) = v \circ b_{(k)}^{j0+1} \circ b_{(k+1)}^{j1} \circ b_{(k+2)}^{j2} \circ \cdots .$$

The point now is that if z is in the $\circ$-ideal generated by some $\mathscr{K}_{m,0}$, then so is $\sigma^k(z)$ for all k. Thus, we can take subsets

$$\mathscr{I}_{r+1,t+1} \subset \{\sigma^k(z) \mid k \in \{0, 1, 2, \ldots\}, z \in \mathscr{I}_r, \dim z < t+1\}$$

$$\mathscr{K}_{r+1,t+1} \subset \{\sigma^k(z) \mid k \in \{0, 1, 2, \ldots\}, z \in \mathscr{K}_r, \dim z < t+1\}$$

satisfying the required properties. This completes the inductive step. □

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